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"Interval LU-fuzzy Arithmetic in the Black and Scholes Option Pricing"

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Interval LU-fuzzy arithmetic in the Black and Scholes option pricing*†

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Abstract

In financial markets people have to cope with a lot of uncertainty while making decisions. Many models have been introduced in the last years to handle vagueness but it is very difficult to capture together all the fundamental characteristics of real markets. Fuzzy modeling for finance seems to have some challenging features describing the financial markets behavior; in this paper we show that the vagueness induced by the fuzzy mathematics (see [5], [7] and [14] for more details) can be relevant in modelling objects in finance (see also [2], [8] and [11]), especially when a flexible parametrization is adopted to represent the fuzzy numbers.

Fuzzy calculus for financial applications requires a big amount of computations and the LU-fuzzy representation produces good results due to the fact that it is computationally fast and it reproduces the essential quality of the shape of fuzzy numbers involved in computations. The paper considers the Black and Scholes option pricing formula, as long as many other have done in the last few years (see [3], [13], [15]).

We suggest the use of the LU-fuzzy parametric representation for fuzzy numbers, introduced in Guerra and Stefanini (see [10]) and improved in Stefanini, Sorini and Guerra (see [12]), in the framework of the Black and Scholes model for option pricing, everywhere recognized as a benchmark; the details of the computations by the interval fuzzy arithmetic approach and an illustrative example are also incuded.

1 Introduction

Recent literature on fuzzy numbers is rich of several approaches to approximate operations between fuzzy numbers. The desirable feature is to preserve the real

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shape of the fuzzy numbers resulting from the operations, without loosing in simplicity and applicability and in goodness of the approximations.

In [10] we introduced a representation of the fuzzy numbers, based on the use of parametrized monotonic functions to model the $\alpha - cuts$ (or the membership functions) of the fuzzy numbers. We call it the LU representation, as it models directly the Lower and the Upper branches of the fuzzy numbers and it uses the parametrization to perform the arithmetic operations and more generally for the fuzzy calculus. The LU-fuzzy numbers can also be viewed as a parametrized extension of the standard LR-fuzzy numbers and are related to this extension by a one-to-one (inverse) correspondence. In [12] we show the advantages of the use of LU-fuzzy numbers in the principal applications of fuzzy calculus: they generalize the LR-fuzzy setting in the direction of the shape preservation but also they allow easy error-controlled approximations in fuzzy calculus.

It is well known in economics and in finance that the application of the option theory, both for the real and for the financial markets, is strongly dependent on the precision of the input data and that in many cases the quality of the information becomes critical to the validity of the results. A suitable methodology to approach this problems is based of the fuzzy calculus as it allows the description of uncertain or imprecise interest rates, volatility, prices, etc in combination with the stochastic (risky) characters of the real world.

On the other hand, the inclusion of vagueness and stochasticity into the models frequently implies massive computations to simulate or perform the calculations.

In section 2 we recall some fundamental properties of fuzzy calculus and we describe the LU model for fuzzy numbers, trying to focus on its advantages both in the flexibility and adaptability to real situations and in the speed of the computations.

In section 3 we apply the LU model to the Black and Scholes fuzzy option pricing formula, in a way similar to the one proposed by Wu in [13] (see also [15] and [3] for recent related aspects) where the interval fuzzy arithmetic setting is adopted.

Some notes and challenging observations conclude the last section.

2 Fuzzy calculus

The relevant aspects of Fuzzy Set Theory have started by its invention due to Zadeh [14] in 1965. In what follows, according to the representation theorem for fuzzy numbers or intervals (see [9]), we use the so called $a - cut$ setting to define a fuzzy number or interval; a fuzzy number (or interval) u is completely determined by any pair $u = (u^-, u^+)$ of functions $u^\pm : [0,1] \longrightarrow R$, defining the end-points of the $a - cuts$, satisfying the three conditions:
(i) $u^- : \alpha \longrightarrow u^-_\alpha \in R$ is a bounded monotonic increasing (non decreasing) left-continuous function $\forall \alpha \in]0,1]$ and right-continuous for $\alpha = 0$;
(ii) $u^+ : \alpha \longrightarrow u^+_\alpha \in R$ is a bounded monotonic decreasing (non increasing) left-continuous function $\forall \alpha \in]0,1]$ and right continuous for $\alpha = 0$;

(iii) $u_\alpha^- \leq u_\alpha^+ \; \forall \alpha \in [0,1]$.

If $u_1^- < u_1^+$ we have a fuzzy interval and if $u_1^- = u_1^+$ we have a fuzzy number; for simplicity we refer to fuzzy numbers or intervals without nominal distinction.

The notation

$$[u]_\alpha = \left[u_\alpha^-, u_\alpha^+\right], \; \alpha \in [0,1] \tag{1}$$

denotes explicitly the $\alpha - cuts$ of u. We can refer to u^- and u^+ as the lower and the upper branches of u, respectively.

The membership function $\mu(x)$ of the fuzzy number u is given by the equations $\mu(u_\alpha^-) = \mu(u_\alpha^+) = \alpha$ for $\alpha \in [0,1]$ i.e.

$$\mu(x) = \begin{cases} 0 \;\text{ if } \; x \notin [u_0^-, u_0^+] \\ \alpha \;\text{ if } \; x = u_\alpha^- \text{ or } x = u_\alpha^+ \text{ for a given } \alpha \in [0,1[\\ 1 \;\text{ if } \; x \in [u_1^-, u_1^+]. \end{cases} \tag{2}$$

The arithmetic operations for two fuzzy numbers $u = (u^-, u^+)$ and $v = (v^-, v^+)$ are all defined in terms of the $\alpha - cuts$ (addition, scalar multiplication, product and division), for $\alpha \in [0,1]$, as:

$$[ku]_\alpha = \left\langle \begin{array}{ll} [ku_\alpha^-, ku_\alpha^+] & \text{if } \; k \geq 0 \\ \left[ku_\alpha^+, ku_\alpha^-\right] & \text{if } \; k \leq 0, \end{array} \right. \tag{3}$$

$$[u+v]_\alpha = \left[u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+\right], \tag{4}$$

$$[uv]_\alpha = [(uv)_\alpha^-, (uv)_\alpha^+]$$

where

$$\begin{cases} (uv)_\alpha^- = \min\{u_\alpha^- v_\alpha^-, u_\alpha^- v_\alpha^+, u_\alpha^+ v_\alpha^-, u_\alpha^+ v_\alpha^+\} \\ (uv)_\alpha^+ = \max\{u_\alpha^- v_\alpha^-, u_\alpha^- v_\alpha^+, u_\alpha^+ v_\alpha^-, u_\alpha^+ v_\alpha^+\} \end{cases} . \tag{5}$$

and, if $0 \notin \left[v_0^-, v_0^+\right]$,

$$\left[\frac{u}{v}\right]_\alpha = \left[\left(\frac{u}{v}\right)_\alpha^-, \left(\frac{u}{v}\right)_\alpha^+\right]$$

where

$$\begin{cases} \left(\frac{u}{v}\right)_\alpha^- = \min\left\{\frac{u_\alpha^-}{v_\alpha^-}, \frac{u_\alpha^-}{v_\alpha^+}, \frac{u_\alpha^+}{v_\alpha^-}, \frac{u_\alpha^+}{v_\alpha^+}\right\} \\ \left(\frac{u}{v}\right)_\alpha^+ = \max\left\{\frac{u_\alpha^-}{v_\alpha^-}, \frac{u_\alpha^-}{v_\alpha^+}, \frac{u_\alpha^+}{v_\alpha^-}, \frac{u_\alpha^+}{v_\alpha^+}\right\}. \end{cases} \tag{6}$$

In a way similar to the arithmetic operations, any real function $f : A \longrightarrow \mathbb{R}$ with $A \subseteq \mathbb{R}^p$ can me extended to the fuzzy numbers. We recall that the fuzzy extension of a continuous function f to a fuzzy argument $u = (u^{(1)}, u^{(2)}, ..., u^{(p)})$ of p fuzzy components $u^{(i)}$ having $\alpha - cuts$ $[u^{(i)}]_\alpha = \left[u_\alpha^{(i)-}, u_\alpha^{(i)+}\right]$, has $\alpha - cuts$

$$\begin{aligned} [f(u)]_\alpha &= [f_\alpha^-(u), f_\alpha^+(u)] \quad \text{with} \\ f_\alpha^-(u) &= \min\left\{f(x_1, ..., x_p) \mid x_i \in [u^{(i)}]_\alpha\right\} \text{ and} \\ f_\alpha^+(u) &= \max\left\{f(x_1, ..., x_p) \mid x_i \in [u^{(i)}]_\alpha\right\}. \end{aligned} \tag{7}$$

If, with respect to all the parameters, f is monotonic increasing we obtain

$$[f(u)]_\alpha = \left[f\left(u_\alpha^{(1)-}, ..., u_\alpha^{(p)-}\right), f\left(u_\alpha^{(1)+}, ..., u_\alpha^{(p)+}\right)\right]$$

while, if f is monotonic decreasing, it holds:

$$[f(u)]_\alpha = \left[f\left(u_\alpha^{(1)+}, ..., u_\alpha^{(p)+}\right), f\left(u_\alpha^{(1)-}, ..., u_\alpha^{(p)-}\right)\right].$$

2.1 LU parametric fuzzy numbers and LU-fuzzy calculus

The LU-representation of the fuzzy numbers is obtained by writing the lower and the upper branches $u^-, u^+ : [0,1] \to R$ as monotonic functions. As mentioned in [10] and [12], a family of monotonic rational or mixed cubic-exponential splines can be used as flexible and easy-to-implement models for the parametrization of the LU-fuzzy numbers. The parameters included in the models (number and position of the nodes, values and slopes) allow a wide range of shapes to be taken into account; further, any given fuzzy number can be approximated to a given prescribed precision at the cost of increasing the number of the nodes. The computational experimentations reported in [10] and the extended applications in [12] suggest that the LU-fuzzy numbers are useful tools in the fuzzy calculus.

In the current application we use two monotonic models: a (2,2)-rational spline given by the ratio of two second degree polynomials) and a mixed cubic-exponential spline.

In the simpler forms, the two (continuous) monotonic branches u^- and u^+ are parametrized by 8 numbers $(u_0^-, \delta_0^-, u_0^+, \delta_0^+; u_1^-, \delta_1^-, u_1^+, \delta_1^+)$ giving the values u_0, u_1 and the slopes δ_0, δ_1 (we omit the superscripts $+$ and $-$ for simplicity) at the extremal points of the $[0,1]$ interval; the two interpolating functions for u^- and u^+ for $\alpha \in [0,1]$ are obtained as functions $s(\alpha)$ satisfying $s(0) = u_0$, $s'(0) = \delta_0$, $s(1) = u_1$, $s'(1) = \delta_1$ (if $u_1 = u_0$ we assume $\delta_0 = \delta_1 = 0$ and a constant $s(\alpha)$):

$$\left\{ \begin{array}{c} s(\alpha) = \frac{p(\alpha)}{q(\alpha)} \\ \text{where} \\ p(\alpha) = (u_1 - u_0)u_1\alpha^2 + (u_1\delta_0 + u_0\delta_1)\alpha(1-\alpha) + (u_1 - u_0)u_0(1-\alpha)^2 \\ q(\alpha) = (u_1 - u_0)\alpha^2 + (\delta_0 + \delta_1)\alpha(1-\alpha) + (u_1 - u_0)(1-\alpha)^2 \end{array} \right. \tag{8}$$

for the (2,2)-rational model, and

$$\left\{ \begin{array}{c} s(\alpha) = u_0 + (u_1 - u_0 - \frac{\delta_0+\delta_1}{v})\alpha^2(3-2\alpha) + \frac{\delta_0}{v} - \frac{\delta_0}{v}(1-\alpha)^v + \frac{\delta_1}{v}\alpha^v, \\ \text{where} \\ v = 1 + \frac{\delta_0+\delta_1}{u_1-u_0} \end{array} \right. \tag{9}$$

for the mixed model.

If the data are increasing (i.e. $u_1 \geq u_0$ and $\delta_0 \geq 0, \delta_1 \geq 0$) then $s(\alpha)$ is increasing for all the values of $\alpha \in [0,1]$; if the data are decreasing (i.e. $u_1 \leq u_0$ and $\delta_0 \leq 0, \delta_1 \leq 0$) then $s(\alpha)$ is decreasing for all the values of $\alpha \in [0,1]$. So, simply by choosing the data in the appropriate way, we can model the lower (increasing) and the upper (decreasing) branches of the fuzzy numbers.

An interesting property of (8) is that its inverse can be computed analytically, while (9) is such that if the data are linear ($\delta_0 = \delta_1 = u_1 - u_0$) or quadratic ($\delta_0 + \delta_1 = 2(u_1 - u_0)$) or polynomial ($\delta_0 + \delta_1 = n(u_1 - u_0)$, $n > 2$) then $s(\alpha)$ is a linear, a quadratic or an $(n+1)$-polynomial (a cubic shape is obtained for $\delta_0 = \delta_1 = \frac{3}{2}(u_1 - u_0)$ or $\delta_0 = \delta_1 = 0$ but $u_1 > u_0$). Note also that in models (8) or (9) any non negative values are admitted for the slopes δ_0 and δ_1 to reproduce non decreasing functions of a very large family (depending on u_0 and $u_1 > u_0$ for the values and on two arbitrary non negative parameters, ranging from zero to infinity, for the slopes); we will use une of the following notations to indicate the spline function s:

$$s(\alpha) = s(\alpha; u, \delta) = s(\alpha; u_0, \delta_0, u_1, \delta_1). \tag{10}$$

The models allows a representation of the $\alpha - cuts$ of the LU-fuzzy number by four pairs of values $u = (u_0^-, \delta_0^-, u_0^+, \delta_0^+; u_1^-, \delta_1^-, u_1^+, \delta_1^+)$: each pair (u_i^*, δ_i^*), with $i \in \{0, 1\}$ and $* \in \{-, +\}$ refers to one of the four interpolating points corresponding to $\alpha_0 = 0$ and to $\alpha_1 = 1$ for the lower and the upper $\alpha - cut$ branches.

To obtain the general (continuous) LU-representation, we introduce a decomposition of the α's into $N+1$ nodes (N subintervals) $0 = \alpha_0 < \alpha_1 < ... < \alpha_N = 1$ and we model each branch u^- and u^+ by a piecewise function of the forms above. Let $(u_i^-, \delta_i^-)_{i=0,1,...,N}$ and $(u_i^+, \delta_i^+)_{i=0,1,...,N}$ be given values of the functions u^- and u^+ and of their first derivatives at the $N+1$ nodes of the decomposition; valid values must satisfy the following conditions:

$$\begin{aligned} u_0^- &\leq u_1^- \leq ... \leq u_N^- \leq u_N^+ \leq u_{N-1}^+ \leq ... \leq u_0^+ \\ \delta_i^- &\geq 0 \text{ and } \delta_i^+ \leq 0. \end{aligned}$$

For all the values of $\alpha \in [0, 1]$ the functions u^- and u^+ are computed on each of the N subintervals of the decomposition with a different version of the mixed spline s: after the transformation $\alpha \in [\alpha_{i-1}, \alpha_i] \longleftrightarrow t_\alpha \in [0, 1]$, i.e. $t_\alpha = \frac{\alpha - \alpha_{i-1}}{\alpha_i - \alpha_{i-1}}$ and $\widetilde{\delta}_i^- = \delta_i^-(\alpha_i - \alpha_{i-1})$, $\widetilde{\delta}_i^+ = \delta_i^+(\alpha_i - \alpha_{i-1})$, we get

$$\begin{aligned} u &= (u_i^-, \delta_i^-, u_i^+, \delta_i^+)_{i=0,1,...,N} \\ &\Updownarrow \\ u_\alpha &= [s(t_\alpha; u_{i-1}^-, \widetilde{\delta}_{i-1}^-, u_i^-, \widetilde{\delta}_i^-), s(t_\alpha; u_{i-1}^+, \widetilde{\delta}_{i-1}^+, u_i^+, \widetilde{\delta}_i^+)]_{i=1,2,...,N}. \end{aligned} \tag{11}$$

For $N \geq 1$, an array of $4\,(N+1)$ parameters is available for the lower branch u_α^- (monotonic increasing) and the upper branch u_α^+ (monotonic decreasing); the simple conditions $\delta_i^- \geq 0$, $\delta_i^+ \leq 0$ and $u_0^- \leq u_1^- \leq ... \leq u_N^- \leq u_N^+ \leq u_{N-1}^+ \leq ... \leq u_0^+$ are required.

The set of LU-fuzzy numbers (for a fixed monotonic-shaped model) is denoted by

$$\mathbb{F}_N = \left\{ (u_i^-, \delta_i^-, u_i^+, \delta_i^+)_{i=0,1,...,N} \mid u_i^- \nearrow, \;\; u_i^+ \searrow, \;\; \delta_i^- \geq 0, \;\; \delta_i^+ \leq 0 \right\}.$$

In particular, corresponding to the nodes of the $\alpha-$decomposition, the membership function of u is given by the relations

$$\mu(u_i^-) = \mu(u_i^+) = \alpha_i \text{ for } i = 0, 1, ..., N$$

and, for the differentiable case, it holds:

$$\mu'(u_i^-) = \frac{1}{\delta_i^-}, \quad \mu'(u_i^+) = \frac{1}{\delta_i^+} \text{ for } i = 0, 1, ..., N.$$

As reported in [10], the membership function $\mu(x)$ can be obtained in two ways:

1. it can be approximated with small errors by the use of the monotonic splines similar to the ones proposed for the LU-representation, i.e. by using a double monotonic representation for the left (increasing) and for the right (decreasing) branches of $\mu(x)$; this is easily obtained by writing the two branches as:

$$\mu(x) = \begin{cases} 0 \quad \text{if } x \notin]u_0^-, u_0^+[\\ s(x; u_{i-1}^-, u_i^-, \frac{u_i^- - u_{i-1}^-}{\delta_{i-1}^-}, \frac{u_i^- - u_{i-1}^-}{\delta_i^-}) & \text{if } x \in [u_{i-1}^-, u_i^-], \; i = 1, 2, ..., N \\ 1 \quad \text{if } x \in [u_1^-, u_1^+] \\ s(x; u_i^+, u_{i-1}^+, \frac{u_{i-1}^+ - u_i^+}{\delta_i^+}, \frac{u_{i-1}^+ - u_i^+}{\delta_{i-1}^+}) & \text{if } x \in [u_i^+, u_{i-1}^+], \; i = 1, 2, ..., N; \end{cases} \tag{12}$$

2. alternatively, we can use equations (2) directly and solve, for example by the basic bisection method, the equations (where $s(\alpha)$ is monotonic)

$$\mu(x) = \alpha \Longleftrightarrow \begin{cases} s(\alpha; u_{i-1}^-, \delta_{i-1}^-, u_i^-, \delta_i^-) = x \text{ for } u_{i-1}^- \leq x \leq u_i^- \;, i = 1, 2, ..., N \\ s(\alpha; u_i^+, \delta_i^+, u_{i-1}^+, \delta_{i-1}^+) = x \text{ for } u_i^+ \leq x \leq u_{i-1}^+ \;, i = 1, 2, ..., N. \end{cases} \tag{13}$$

For the particular (2,2) rational spline (8) it is easy to compute the analytic inverse of the spline $s(\alpha)$ by solving the following equation of second degree, to obtain $\alpha = s^{-1}(x) \in [0, 1]$ for $x \in [u_0, u_1]$ or for $x \in [u_1, u_0]$:

$$p(\alpha) - q(\alpha)x = 0$$

i.e.

$$(u_1 - u_0)(u_1 - x)\alpha^2 + [(u_1 - x)\delta_0 + (u_0 - x)\delta_1]\alpha(1-\alpha) + (u_1 - u_0)(u_0 - x)(1-\alpha)^2 = 0.$$

If we define $a_x = (u_1 - u_0)(u_1 - x) \geq 0$, $b_x = (u_1 - x)\delta_0 + (u_0 - x)\delta_1$ and $c_x = (u_1 - u_0)(u_0 - x) \leq 0$, then the inverse spline is given by:

$$s^{-1}(x) = \frac{2c_x - b_x \pm \sqrt{b_x^2 - 4a_x c_x}}{2(a_x + c_x - b_x)} \tag{14}$$

where we select the $+$ or the $-$ in the numerator for which the value is in $[0, 1]$ (note that it is unique and that $b_x^2 - 4a_x c_x \geq 0 \; \forall x$).

It follows that the membership function of u, according to (2), is obtained as

$$\mu(x)=\begin{cases}0 \text{ if } x\notin[u_0^-,u_0^+]\\ s^{-1}(x;u^-,\delta^-) \text{ if } u_0^-\le x\le u_1^-\\ s^{-1}(x;u^+,\delta^+) \text{ if } u_1^+\le x\le u_0^+\\ 1 \text{ if } x\in[u_1^-,u_1^+].\end{cases} \tag{15}$$

where $s^{-1}(x;u^-,\delta^-)$ and $s^{-1}(x;u^+,\delta^+)$ denote the inverses of the splines obtained for the lower (with the parameters having the superscript $-$) and the upper (with the parameters having the superscript $+$) branches, respectively.

The arithmetic operations, the Zadeh's fuzzy extensions, the fuzzy integral and derivative and other elements of fuzzy calculus can be defined in F_N.

As an example, the fuzzy multiplication is obtained by a relatively simple algorithm: denote $uv=w=\left(w_i^-,f_i^-,w_i^+,f_i^+\right)_{i=0,1,\dots,N}$, and

$$(uv)_i^-=\min\{u_i^-v_i^-,u_i^-v_i^+,u_i^+v_i^-,u_i^+v_i^+\} \tag{16}$$

$$(uv)_i^+=\max\{u_i^-v_i^-,u_i^-v_i^+,u_i^+v_i^-,u_i^+v_i^+\}; \tag{17}$$

let (p_i^-,q_i^-) be the pair associated to the combination of + and - of $u_i^{\pm}v_i^{\pm}$ giving the minimum for $(uv)_i^-$ in (16), and similarly let (p_i^+,q_i^+) be the pair associated to the combination of + and - of $u_i^{\pm}v_i^{\pm}$ giving the maximum for $(uv)_i^+$ in (17), then (for the values and the slopes of $w=uv$)

$$\begin{cases} w_i^-=u_i^{p_i^-}v_i^{q_i^-} \quad , \quad w_i^+=u_i^{p_i^+}v_i^{q_i^+}\\ f_i^-=\delta_i^{p_i^-}v_i^{q_i^-}+u_i^{p_i^-}e_i^{q_i^-} \quad , \quad f_i^+=\delta_i^{p_i^+}v_i^{q_i^+}+u_i^{p_i^+}e_i^{q_i^+}.\end{cases}$$

To obtain the fuzzy extension of a real function $f:\mathbb{R}\longrightarrow\mathbb{R}$ (we consider here for simplicity the simple case where f is monotonic) let's introduce the following notation:

$$p_\alpha^-,p_\alpha^+\in\{-,+\} \text{ and}$$

$$p_\alpha^-=\begin{cases}- & \text{if} \quad \min\{f(u_\alpha^-),f(u_\alpha^+)\}=f(u_\alpha^-)\\ + & \text{if} \quad \min\{f(u_\alpha^-),f(u_\alpha^+)\}=f(u_\alpha^+)\end{cases}$$

$$p_\alpha^+=\begin{cases}- & \text{if} \quad \max\{f(u_\alpha^-),f(u_\alpha^+)\}=f(u_\alpha^-)\\ + & \text{if} \quad \max\{f(u_\alpha^-),f(u_\alpha^+)\}=f(u_\alpha^+)\end{cases}$$

(where we simplify $p_{\alpha_i}^{\pm}\equiv p_i^{\pm}$, $i=0,1,\dots,N$, in the points of the $\alpha-$decomposition).

So, we have $f(u)_\alpha^-=f\left(u_\alpha^{p_\alpha^-}\right)$ and $f(u)_\alpha^+=f\left(u_\alpha^{p_\alpha^+}\right)$.

If X is the LU-fuzzy number $X=\left(x_i^-,\delta_i^-,x_i^+,\delta_i^+\right)_{i=0,1,\dots,N}$ then its image $f(X)$ is

$$f(X)=\left(f(x_i^{p_i^-}),f'(x_i^{p_i^-})\delta_i^{p_i^-},f(x_i^{p_i^+}),f'(x_i^{p_i^+})\delta_i^{p_i^+}\right)_{i=0,1,\dots N}.$$

An LU-fuzzy calculator, implementing the basic fuzzy arithmetic operations and calculus by an interactive desk-top easy-to-use software is available by contacting the authors[1].

[1] L. Sorini - L. Stefanini, An LU-fuzzy calculator for the basic fuzzy calculus, EMS working

3 LU-fuzzy Black-Scholes

Many empirical studies have shown that some hypothesis of the Black-Scholes (B-S) model for european options (introduced in [1]) do not reflect in a satisfactory way the behavior of financial markets. In particular the risk-free rate r is considered constant but in the real world it varies in an imprecise way. Moreover, a big amount of recent financial literature is devoted to the volatility modeling and many different approaches have been studied; we believe that the fuzzy modeling of volatility can contribute in the research of a benchmark volatility model, especially when an easy model to implement is adopted.

The B-S formulation consists in determining the current value C_t of an european call option of a derivative security having known current stock price and exercise price at maturity time T. The volatility and the so called risk-neutral interest rate are the other parameters by which the B-S formula is composed. All these parameters, with the possible exception of the strike price and of the time to maturity, can be modelled as fuzzy numbers. In our calculations, the stock price, the strike price, the volatility and the interest rate are handled as fuzzy numbers due to their imprecise character. The strike price and the time to maturity are supposed to be real numbers because they have this characteristics in the real world.

The general procedure for the interval fuzzy arithmetic B-S model is very similar to that described by Wu (see [13]); we focus on the alternative way of representing the fuzzy numbers, the LU representation illustrated in the preceeding section, and on its advantages in producing quickly the final solution.

Denote, as usual, by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$$

the cumulative standard normal function.

It is an increasing function with derivative

$$\Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

To introduce the standard Black Scholes formula for the European put/call options (without dividends), denote

$$D_1(S, K, r, \sigma, \tau) = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

and

$$D_2(S, K, r, \sigma, \tau) = D_1(S, K, r, \sigma, t) - \sigma\sqrt{\tau}.$$

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We also write

$$D_1 = \frac{\ln\left(\frac{S}{K}\right) + r\tau}{\sigma\sqrt{\tau}} + \frac{\sigma}{2}\sqrt{\tau}$$

$$D_2 = \frac{\ln\left(\frac{S}{K}\right) + r\tau}{\sigma\sqrt{\tau}} - \frac{\sigma}{2}\sqrt{\tau}.$$

If S_t is the (current) stock price at time $t \in [0, T]$ (where T is the time to maturity), K is the exercise price (strike price), r is the interest rate (continuously compounded), σ is the (standard deviation) volatility, then the price C_{T-t} of the corresponding European call option at time t with maturity T is given by (here $\tau = T - t$):

$$C_\tau = S_t \Phi\left(D_1\left(S_t, K, r, \sigma, \tau\right)\right) - Ke^{-r\tau}\Phi\left(D_2\left(S_t, K, r, \sigma, \tau\right)\right) \tag{18}$$

and the price P_τ of the corresponding European put option at time t (with the same expiry date T and strike price K) is given by:

$$P_\tau = C_\tau + Ke^{-r\tau} - S_t. \tag{19}$$

It is also important to compute a probability density function for the option value C_t, given by (log-normal type):

$$D\left(C_\tau\right) = \frac{a}{2\pi} e^{-\frac{1}{2}b^2} \tag{20}$$

where

$$a = \frac{e^{r\tau}}{\sigma\tau\left(C_\tau e^{r\tau} + S_t\right)}$$

and

$$b = \frac{\ln\left(\frac{C_\tau e^{r\tau} + K}{S}\right) - \left(r - \frac{\sigma^2}{2}\right)\tau}{\sqrt{\sigma\tau}}.$$

3.1 Application of the LU-fuzzy representation

The fuzzification of (18) and (19) assumes that K, T are given crisp numbers and $r, \sigma, \{S_t; t \in [0, T]\}$ are given fuzzy numbers; in the LU-fuzzy setting, denote:

$$r = \left(r_i^-, \delta r_i^-, r_i^+, \delta r_i^+\right)_{i=0,1,\dots,N}$$

$$\sigma = \left(\sigma_i^-, \delta\sigma_i^-, \sigma_i^+, \delta\sigma_i^+\right)_{i=0,1,\dots,N}$$

$$S_t = \left(S_{t,i}^-, \delta S_{t,i}^-, S_{t,i}^+, \delta S_{t,i}^+\right)_{i=0,1,\dots,N}$$

and $0 = \alpha_0 < \alpha_1 < \dots < \alpha_N = 1$.

To compute the fuzzy extensions of (18), (19) and (20), observe first that all the involved quantities are non negative crisp or fuzzy numbers so that all the arithmetic operations are simplified; the procedure works as follows:

Step 1. Compute the fuzzy extension of the function

$$S_t \quad \rightarrow \quad \ln\left(\frac{S_t}{K}\right) \tag{21}$$

$(S_t$ fuzzy, K crisp, t given).

The strike price K is a crisp positive quantity and we use the scalar multiplication

$$\frac{S_t}{K} = \left(\frac{S_{t,i}^-}{K}, \frac{\delta S_{t,i}^-}{K}, \frac{S_{t,i}^+}{K}, \frac{\delta S_{t,i}^+}{K}\right)_{i=0,1,\ldots,N}$$

as the function $x \rightarrow \ln(x)$ is increasing, the fuzzy extension of (21) is the following

$$\ln\left(\frac{S_t}{K}\right) = \left(\ln\left(\frac{S_{t,i}^-}{K}\right), \frac{\delta S_{t,i}^-}{S_{t,i}^-}, \ln\left(\frac{S_{t,i}^+}{K}\right), \frac{\delta S_{t,i}^+}{S_{t,i}^+}\right)_{i=0,1,\ldots,N}$$

Denote this fuzzy number by:

$$L_t = \left(L_{t,i}^-, \delta L_{t,i}^-, L_{t,i}^+, \delta L_{t,i}^+\right)_{i=0,1,\ldots,N}$$

Step 2. Compute $D_{1,t}$ and $D_{2,t}$ by extending the fuzzy functions

$$D_{1,t} : (S_t, r, \sigma) \longrightarrow \frac{\ln\left(\frac{S_t}{K}\right) + r\tau}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2} \tag{22}$$

and

$$D_{2,t} : (S_t, r, \sigma) \longrightarrow \frac{\ln\left(\frac{S_t}{K}\right) + r\tau}{\sigma\sqrt{\tau}} - \frac{\sigma\sqrt{\tau}}{2} \tag{23}$$

as functions of S_t, r and σ for given (crisp) K, T, t where $\tau = T - t$.

If

$$A_t = \ln\left(\frac{S_t}{K}\right) + r\tau$$

we denote

$$A_t = \left(A_{t,i}^-, \delta A_{t,i}^-, A_{t,i}^+, \delta A_{t,i}^+\right)_{i=0,1,\ldots,N}$$

in order to obtain:

$$A_{t,i}^- = \left(L_{t,i}^- + r_i^-\left(T-t\right), \delta L_{t,i}^- + \delta r_i^-\left(T-t\right), L_{t,i}^+ + r_i^+\left(T-t\right), \delta L_{t,i}^+ + \delta r_i^+\left(T-t\right)\right).$$

The fuzzy extensions of $D_{1,t}$ and $D_{2,t}$ in the LU-representation can be written as:

$$D_{1,t} = \left(D_{1,t,i}^-, \delta D_{1,t,i}^-, D_{1,t,i}^+, \delta D_{1,t,i}^+\right)_{i=0,1,\ldots,N}$$
$$D_{2,t} = \left(D_{2,t,i}^-, \delta D_{2,t,i}^-, D_{2,t,i}^+, \delta D_{2,t,i}^+\right)_{i=0,1,\ldots,N}$$

and the values of the parameters are obtained by applying the extension principle to the functions (22) and (23). We omit here the tedious details for brevity;

observe only that, with respect to $x = \sigma\sqrt{T-t}$, $D_{1,t}$ and $D_{2,t}$ have the following form:

$$D_{1,t} : x \longrightarrow \frac{A_t}{x} + \frac{x}{2}, \qquad D_{2,t} : x \longrightarrow \frac{A_t}{x} - \frac{x}{2}$$

and their fuzzy extensions require to distinguish the two cases $A_t \geq 0$ and $A_t \leq 0$; in fact, $D_{1,t}$ is increasing if $A_t \leq 0$ but has a global minimum point at $x = \sqrt{2A_t}$ if $A_t > 0$; analogously, $D_{2,t}$ is decreasing if $A_t \geq 0$ but has a global maximum point at $x = \sqrt{(-2A_t)}$ if $A_t < 0$.

Step 3. Compute $\Phi(D_{1,t})$ and $\Phi(D_{2,t})$ as the fuzzy extension of the cumulative standard normal function.

As the function $x \to \Phi(x)$ is differentiable and monotonic increasing, it is immediate to obtain its fuzzy extension $\Phi(D_{k,t})$ $k = 1, 2$:

$$\Phi_{k,t} = \left(\Phi\left(D^-_{k,t,i}\right), \delta D^-_{k,t,i}\Phi'\left(D^-_{k,t,i}\right), \Phi\left(D^+_{k,t,i}\right), \delta D^+_{k,t,i}\Phi'\left(D^+_{k,t,i}\right)\right) \tag{24}$$

To compute the (integral) cumulative function $\Phi(x)$ we use a well known approximation[2].

Step 4 Compute the fuzzy extension of

$$r \quad \longrightarrow \quad e^{-r(T-t)} \tag{25}$$
$$(T \text{ and } t \text{ crisp, } r \text{ fuzzy})$$

The fuzzy extension of this differentiable decreasing function has the following LU-representation

$$e^{-r(T-t)} = \left(\exp\left(-r_i^+(T-t)\right), -(T-t)\,\delta r_i^+ \exp\left(-r_i^+(T-t)\right),\right.$$
$$\left.\exp\left(-r_i^-(T-t)\right), -(T-t)\,\delta r_i^- \exp\left(-r_i^-(T-t)\right)\right)_{i=0,1,\ldots,N}$$

denote it by

$$E_t = \left(E^-_{t,i}, \delta E^-_{t,i}, E^+_{t,i}, \delta E^+_{t,i}\right)_{i=0,1,\ldots,N}$$

Step 5 Compute C_t by the arithmetic operations in (18).

If the LU-representations of the fuzzy extension of C_τ, P_τ and $D(C_\tau)$ are denoted by

$$C_\tau = \left(C^-_{\tau,i}, \delta C^-_{\tau,i}, C^+_{\tau,i}, \delta C^+_{\tau,i}\right)_{i=0,1,\ldots,N}$$
$$P_\tau = \left(P^-_{\tau,i}, \delta P^-_{\tau,i}, P^+_{\tau,i}, \delta P^+_{\tau,i}\right)_{i=0,1,\ldots,N}$$
$$D_\tau = \left(D^-_{\tau,i}, \delta D^-_{\tau,i}, D^+_{\tau,i}, \delta D^+_{\tau,i}\right)_{i=0,1,\ldots,N}$$

we easily obtain (due to the fact that all quantities are positive) the following

$$C^-_{\tau,i} = S^-_{t,i}\Phi^-_{1,t,i} - KE^+_{t,i}\Phi^+_{2,t,i}$$

$$\delta C^-_{\tau,i} = \delta S^-_{t,i}\Phi^-_{1,t,i} + S^-_{t,i}\delta\Phi^-_{1,t,i} - K\left(\delta E^+_{t,i}\Phi^+_{2,t,i} + E^+_{t,i}\delta\Phi^+_{2,t,i}\right)$$

[2] For more details see W.H. Press *et al.* Numerical Recipes in C: the Art of Scientific Computing, Cambridge University Press, second edition, 1992.

$$C^+_{\tau,i} = S^+_{t,i}\Phi^+_{1,t,i} - KE^-_{t,i}\Phi^-_{2,t,i}$$

$$\delta C^+_{\tau,i} = \delta S^+_{t,i}\Phi^+_{1,t,i} + S^+_{t,i}\delta\Phi^+_{1,t,i} - K\left(\delta E^-_{t,i}\Phi^-_{2,t,i} + E^-_{t,i}\delta\Phi^-_{2,t,i}\right)$$

The computation of the fuzzy price P_τ in (19) and of the probability D_τ in (20) can be made in a similar way as for C_τ. We only note that equation (19) defines P_τ as the Hukuhara difference:

$$P_\tau = C_\tau + Ke^{-r\tau} \overset{h}{-} S_t$$

where $\overset{h}{-}$ is defined by the equality (if the fuzzy Z exists)

$$Z = X \overset{h}{-} Y \Longleftrightarrow Z + Y = X.$$

Introducing the term $E_t = Ke^{-r(T-t)}$ the following equalities hold $\forall\alpha \in [0,1]$:

$$\begin{cases} P^-_{\tau,\alpha} + S^-_{t,\alpha} & = & C^-_{\tau,\alpha} + E^-_{t,\alpha} \\ P^+_{\tau,\alpha} + S^+_{t,\alpha} & = & C^+_{\tau,\alpha} + E^+_{t,\alpha} \end{cases}$$

that are equivalent to the following (after (18) and (24)):

$$\begin{cases} P^-_{\tau,\alpha} & = & S^-_{t,\alpha}\left(\Phi^-_{1,t,\alpha} - 1\right) + E^-_{t,\alpha} - E^+_{t,\alpha}\Phi^+_{2,t,\alpha} \\ P^+_{\tau,\alpha} & = & S^+_{t,\alpha}\left(\Phi^+_{1,t,\alpha} - 1\right) + E^+_{t,\alpha} - E^-_{t,\alpha}\Phi^-_{2,t,\alpha}. \end{cases}$$

The final step of our computation is the construction of the membership function by inverting the LU representation. As illustrated in the previous section, depending on the adopted model for the spline, we can use the analytic inverses as in (14) and (15) for the (2,2)-rational spline, or one of the two methods (12) or (13) for the mixed spline model.

3.2 Computational results

As an example of the illustrated procedures, we consider the same evaluation reported by Wu ([13]). In our case, the complexity of the computations is extremely reduced: as we will see, only 5 points in the $\alpha-$decomposition ($N = 4$) give an error less than 0.004% (with a precision between the fourth and the fifth decimal place) for all the values of the membership greater than $\alpha = 0.5$ (note that, in the financial applications, usual values of interest for α are between 0.9 and 1.0). In the framework of the LU-fuzzy calculus, the results are obtained in a much simpler way and the calculation of the membership function of the involved quantities makes also possible their use in further fuzzy calculations. In particular, the problem to find the value of α such that $\mu(C) = \alpha$ (where μ is the membership function) can be solved in closed form and does not require additional computational effort.

In our computations, the results are obtained by the LU-fuzzy decomposition with $N = 1, 2, 4$ and 10 and by an exact procedure in 101 uniform α points. The approximation with $N = 10$ is exact up to 8 decimal places and is not reported further.

The same data as in ([13]) are used: $K = 30$, $T = 0.25$, $t = 0$, $\tau = T - t = 0.25$, $S_0 = (32, 33, 34)$, $r = (0.048, 0.05, 0.052)$ and $\sigma = (0.08, 0.1, 0.12)$ (triangular and symmetric fuzzy numbers are written in the form $u = (u_0^-, u_1^- = u_1^+, u_0^+))$.

The following tables give the LU-representation for the data S_0, r and σ for the first $\alpha-$ decomposition ($N = 1$):

$u = S_0$:

α_i	u_i^-	du_i^-	u_i^+	du_i^+
0	32	1	34	-1
1	33	1	33	-1

$u = r$:

α_i	u_i^-	du_i^-	u_i^+	du_i^+
0	0.048	0.002	0.052	-0.002
1	0.05	0.002	0.05	-0.002

$u = \sigma$:

α_i	u_i^-	du_i^-	u_i^+	du_i^+
0	0.08	0.02	0.12	-0.02
1	0.1	0.02	0.1	-0.02

Figure 1. illustrates the calculated exact and approximated membership functions for the α values of interest. Note that the approximations with $N = 2$ and $N = 4$ are graphically coincident with the exact solution.

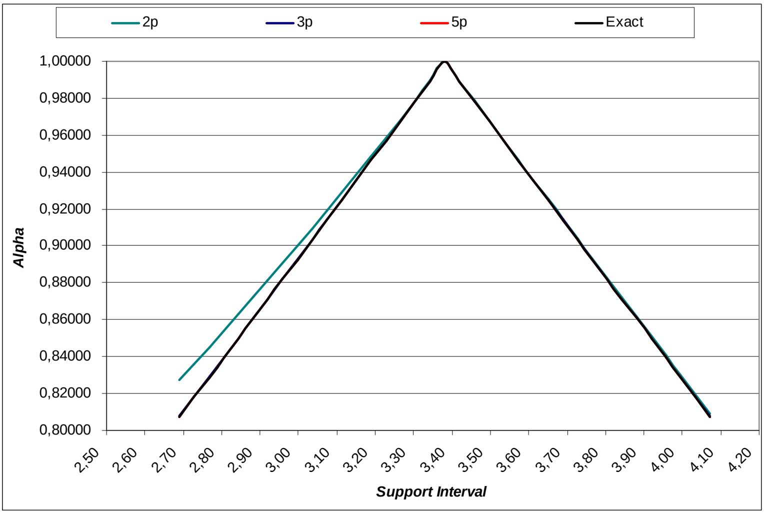

Figure 1: C_τ membership functions for different approximations

The obtained LU-representation of C_τ is reported for the two $\alpha-$ decompositions (N = 2, 4) in the following table:

$u = C_\tau$:	α_i	u_i^-	du_i^-	u_i^+	du_i^+
N=2	0	-0.7549	5.0881	7.5312	-5.1247
	0.5	1.5173	4.0371	5.2478	-4.0486
	1	3.3813	3.5673	3.3813	-3.5673
N=4	0	-0.7549	5.0881	7.5312	-5.1247
	0.25	0.4479	4.5355	6.3212	-4.5570
	0.5	1.5173	4.0371	5.2478	-4.0486
	0.75	2.4792	3.6903	4.2839	-3.6951
	1	3.3813	3.5673	3.3813	-3.5673

The percentage error of the 5-point approximation is reported in figure 2. (here a zero error means exact to the fifth decimal).

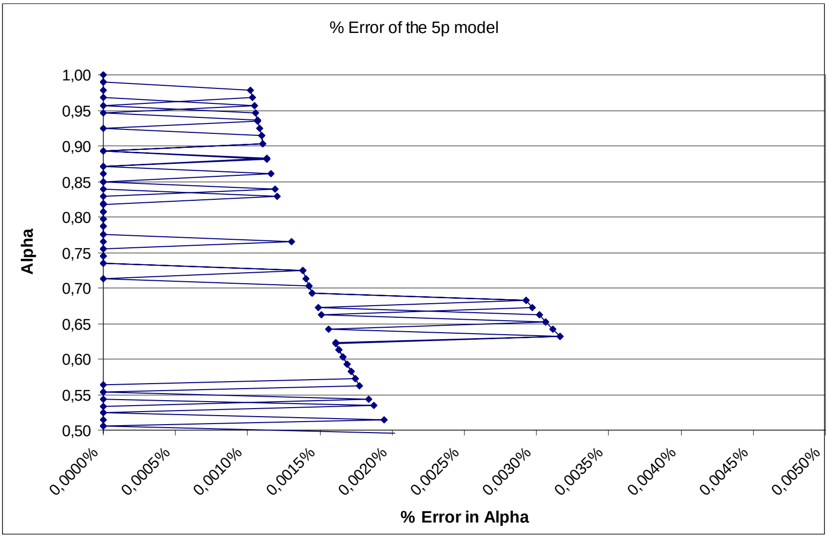

Figure 2: Percentage error in the 5 points approximation

The remaining three figures 3. 4. and 5. illustrate the membership functions of the fuzzy option value C_τ, the fuzzy price value P_τ and the fuzzy probability density $D(C_\tau)$ obtained by the mixed spline model.

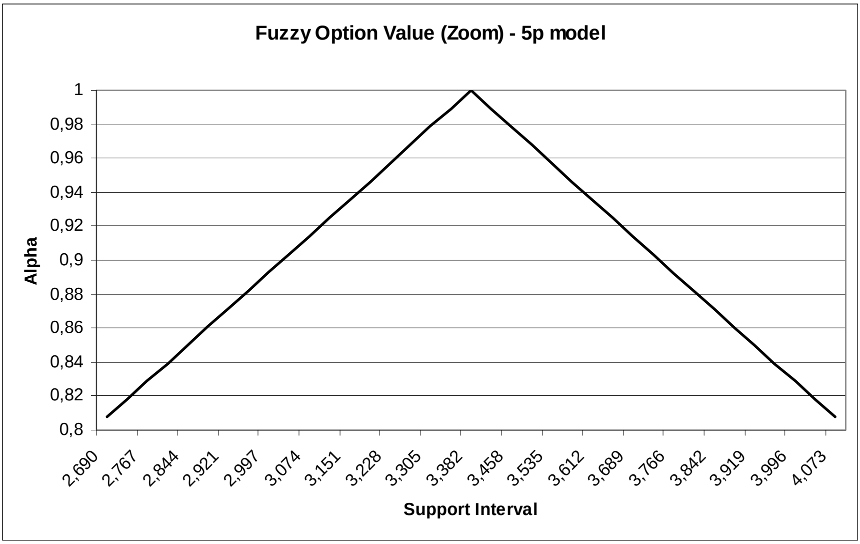

Figure 3: Fuzzy call Value

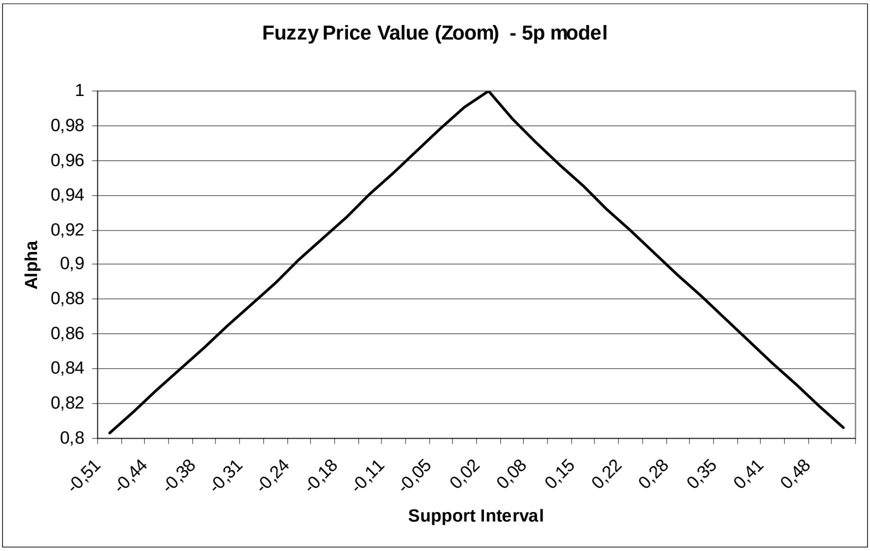

Figure 4: Fuzzy put Price

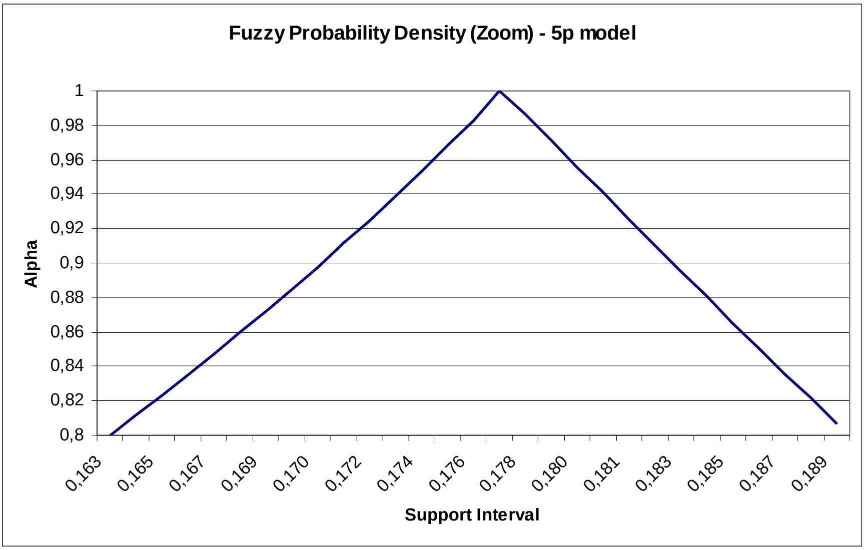

Figure 5: Fuzzy Probability Density

It appears that the LU parametric representation and the corresponding calculus maintain the shape of the fuzzy numbers and are precise also when few nodes are considered. This means a relevant saving in the computations and the advantage of obtaining the membership functions which can be used for further fuzzy operations.

4 Conclusions

We study the peculiarities of LU parametric representation in the fuzzy version of Black-Scholes model. In details we show the advantages of LU-fuzzy numbers when finding the European call option price where some input variables are taken as fuzzy numbers.

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